Enhanced Indexing and Selectivity Theory

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Abstract

Investment managers that believe to have skill must choose some fraction of stocks in a benchmark to hold. Recent theory predicts that the optimal percentage of holdings for a manager with skill is between 50 and 80 percent of the benchmark. This theory requires a number of assumptions. Using simulations, we relax some of the assumptions to examine if the theory still holds. We find that, for the most part, the theory holds when the assumptions are relaxed. We also extend the analysis from a one-period horizon to a multi-period horizon using the actual returns of stocks in the S&P 500.

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Past research on the topic of active versus passive investment management has broadly focused on the returns of active managers: whether there is evidence of outperformance in excess of the manager's fees and expenses (Fama & French (2010), Berk & van Binsbergen (2015)), and what portfolio characteristics tend to correspond with outperformance (Cremers & Pareek (2016)). For managers pursuing a near-passive or enhanced index approach to investing, the Information Ratio (IR) of their portfolio is the relevant metric to assess performance. Recent theory (Bolshakov & Chincarini (2019)) provides a novel theoretical approach to determining the optimal number of stocks a skillful manager should hold from their benchmark in order to maximize their IR. The theory focuses on the ratio of the number of stocks selected by the portfolio manager divided by the number of benchmark constituents and labels this as the "selectivity ratio" of the manager. The authors find that depending on whether stocks are purchased into the portfolio in a single, one-time, bulk selection or by sequential, discrete additions, the manager's IR will be optimized at either a 50% selectivity ratio or at an 80% selectivity ratio, respectively. The theoretical approach is novel and provocative, but introduces certain simplifying assumptions to derive their results about optimal selectivity.

This paper uses Monte Carlo simulations to relax the assumptions in that paper to determine whether the theory holds even when the simplistic assumptions are relaxed. Generally, the theory is quite robust even in the presence of relaxing the assumptions. In addition, the use of simulations allows the work of Bolshakov and Chincarini (2019) to be extended to a more realistic dynamic investment environment, where portfolio managers are able to select stocks every month, rather than for one investment period. This expanded dynamic environment suggests that the optimal range of selectivity for a portfolio manager, based on various assumptions in the dynamic case, is somewhere between 50% and 80% of the manager's benchmark.

The paper is organized as follows: first we briefly review the selectivity theory; then we discuss the data and methodology used in this paper; we then report the results of the simulations by relaxing the theoretical assumptions; we then discuss the dynamic investment simulations and results; and we then conclude.

Theory

Bolshakov & Chincarini (2019) consider the case of a skillful portfolio manager seeking to maximize their IR by choosing a portfolio selectivity ratio to target prior to forming their portfolio. The manager's skill is described as a probability weighting advantage to choosing outperforming stocks ("winners") versus underperforming stocks ("losers"). For a skillful manager, this probability advantage is encompassed in ω (i.e. omega), which is greater than 1 for a skilled manager and equal to one for an unskilled manager. Bolshakov and Chincarini (2019) brought this entirely new approach to the portfolio management literature that involves the rarely mentioned Fisher and Wallenius Noncentral Hypergeometric distributions. An ω greater than one means the portfolio manager has a higher likelihood of choosing a winner stock rather than a loser stock.[1] The theory predicts that when a portfolio manager selects stocks in bulk manner, that is, they decide the percentage of stocks that they wish to buy of the benchmark in advance, the process is governed by the Fisher Noncentral Hypergeometric Distribution (FNCH) and it will be optimal for managers to select 50% of the benchmark. When a portfolio manager selects stocks in a sequential manner, then the process is governed by the Wallenius Noncentral Hypergeometric Distribution (WNCH). In this case, the theory suggests that the portfolio manager should choose approximately 80% of the investment universe if such a manager has skill.

The theory makes four simplifying assumption to arrive at these conclusions. The first assumption is that half of the stocks will outperform the index over some period in the future and half will underperform. The second assumption is that the performance of each stock over the investment period can be expressed as a binary event. That is, either it beats the index or it does not. This implicitly assumes that the magnitude of the performance does not matter. The third assumption is that the portfolio manager has a constant ability of picking winner stocks over loser stocks, regardless of how many stocks have been chosen. The fourth assumption is that the benchmark and portfolio are equally weighted. Finally, the theory takes place in a one-period investment horizon, just like the CAPM (Sharpe (1964)),

[1] The parameter ω is not equal to the probability of choosing a winner stock, but it's related. We chose this parameter, since it is standard when using the Fischer and Wallenius Noncentral Hypergeometric distributions, which are relevant to this type of portfolio manager selection.

however, in this paper, we will also present results for dynamic choice and a multi-period horizon.

Data and Methodology

In order to relax the assumptions of the theoretical work, we make use of simulations to examine the behavior of the optimal portfolios and selectivity. Relaxing some of the assumptions requires the use of actual real stock market data. In our work, we make use of the S&P 500 stock return data and market capitalization data from December 31, 1988 to December 31, 2018. We used monthly total stock return data that was obtained from S&P Global. We also obtained the monthly constituents of the S&P 500 from that same database. Over the entire period, company data might be missing for a particular month due to inaccurate data, bankruptcy, delisting, merger and acquisitions, or a host of other corporate actions. We worked closely with the Standard and Poor's team to make sure that our data was void of survivorship bias and, to the extent possible, that our total returns for any stock was accurate, even if a corporate action led to the stock no longer being traded.

The Monte Carlo simulations vary in each simulation in which we relax an assumption of the model, however, the general process is the same. Given a set of parameters and/or real data, we simulate the behavior of an individual portfolio manager in selecting a percentage of the underlying benchmark. In the case of this paper, our benchmark is the S&P 500, and hence the benchmark always has 500 stocks. Thus, we choose every selectivity ratio from 10% (50 stocks) to 90% (450 stocks) in 0.2% intervals, thus replicating a total of 401 selectivity ratios. For each selectivity ratio, we allow the portfolio manager with skill level, ω, to select N_S stocks for that particular selectivity ratio. Thus, for a ten percent selectivity ratio, it would be 50 stocks. For the single-horizon simulations, we do this 100,000 times resulting in 100,000 portfolios chosen with the particular parameters. For the dynamic, multi-period portfolios, we do this 10,000 times resulting in 10,000 portfolios chosen with the particular parameters.[2] We then stored all of the resulting information of that portfolio, including,

[2]The reason we limited the dynamic simulations to just 10,000 is due to the time-intensive nature of the simulations using multiple computers. The qualitative results of the simulations would not change if we did

average return, standard deviation of returns, tracking error, and most importantly the information ratio. We did this separately for the bulk selection method (FNCH) and the sequential selection method (WNCH).[3] Finally, in order to plot or analyze the information ratios based on different selectivity ratios, we took the average for each selectivity ratio across different simulations.

Examination of Assumptions

Baseline Model

Before relaxing the assumptions, we use the Monte Carlo simulations to replicate the base case presented in Bolshakov and Chincarini (2019). In that paper, the authors assumed that the benchmark was composed of 50% winner stocks and 50% loser stocks. They also assumed that the returns of good stocks was 10%, while the returns of bad stocks was -10%. Figure 1 shows the average information ratios from the 100,000 simulations along with the theoretical prediction. As would be expected, the simulations correspond to the theoretical predictions in the baseline case. The figure also shows a fourth-degree polynomial function fit through the simulated IRs.[4] The optimal selectivity ratio based on the polynomial function is marked with a vertical, black line at 79.0%. At that level of selectivity the resulting Information Ratio is 0.855.

[**INSERT FIGURE 1 ABOUT HERE**]

Concentration of Winners

One of our assumptions was that 50% of the stocks in the index would outperform and 50% would underperform in a given period. This assumption is simplistic and does not take

more simulations.

[3] Appendix A describes the simulation process for the Fisher distribution, which is more complicated than the Wallenius from a simulation point of view.

[4] We estimate a fourth degree polynomial using linear regression of the form, $y = ax + bx^2 + cx^3 + dx^4$. The fitted polynomial function achieves a correlation of 0.9997 with the theoretical IRs from the WNCH distribution. Also, since the benchmark return is by construction 0%, each IR is calculated as the ratio of expected returns to standard deviation of returns for simulations at a given selectivity ratio.

into account the positive skewness of stock returns since the upside potential of a stock is unlimited, while the downside loss of a stock is limited. Thus, there could be less winner stocks than loser stocks. In this simulation, we keep all other assumptions the same, however we vary the percentage of winners in the actual universe from 20% to 50%.[5]

[INSERT TABLE 1 ABOUT HERE]

Table 1 shows the benchmark return, the optimal selectivity ratio, the expected return at the optimal selectivity ratio, and the information ratio at the optimal selectivity ratio for different proportions of winner stocks in the benchmark universe. As in the theoretical model for the sequential selection method and the baseline case, the optimal selectivity is around 80% of the stocks in the universe. Decreasing the number of winners from 50% to a more realistic 48% does not alter the optimal ratio, and only slightly diminishes the IR achieved at that point. Even more extreme scenarios, where there are only 20% or 30% winners, only slightly shifts the optimal selectivity point towards holding more stocks. The maximum achievable IR shrinks more drastically though; as winners become scarcer within the benchmark, the manager's skill in selecting winners has fewer opportunities to display itself. In the case where winners make up 30% of the benchmark, small selectivity ratio changes are so immaterial that the original 79.0% point shows a higher IR within its 100,000 simulations than the polynomial-fitted optimal location of 79.4% does.[6]

One may also notice the negative expected return values. This is because with a few amount of winners in the universe, the winner stock returns of 10% are overwhelmed by the loser stock returns of -10%. Thus, the benchmark return and the portfolio return are both negative given our parameters.

[5] To compute IRs for these simulations the numerator takes the difference between expected portfolio return and the constructed benchmark return. The denominator retains its prior definition as the cross-sectional standard deviation of returns within a given selectivity ratio, since the benchmark return is constant across all simulations.

[6] This is just the approximation error from fitting a polynomial function to a series of points.

Binary Return Distribution

Our second assumption was that the magnitude of the returns are not important. This is surely not a fair assumption, since a very good stock pick with a very high return can afford a manager many bad picks of lower magnitude poor returns. This symmetry is unrealistic, thus we use the actual distribution of stock returns, while holding all else constant to examine the impact of this assumption. Figure 2 shows a time-series, cross-sectional average of stock return Z-scores from the S&P 500.[7] Figure 2 illustrates what we already know. Stock returns are not binary and vary substantially within a stock universe.

[INSERT FIGURE 2 ABOUT HERE]

At this point, we use this distribution of stock returns in our simulations to investigate the optimal selectivity ratio. Thus, in every simulation, a portfolio manager selects a certain number of stocks. If the manager picks n_w winner stocks in his basket of n stocks, then we randomly assign a Z-score to each winner stock based on one of the winner Z-scores in Figure 2.[8] We do the same with his loser picks. We then compute the information ratios and plot them against selectivity ratios.

Figure 3 shows the results of the simulations for the sequential selection method. Once again, the maximum IR is achieved very close to the theoretical selectivity ratio of 79.0%. However the maximum IR value declines by 27% compared to the Base Case, from 0.855 to 0.624, as random luck now plays a larger part in the manager's posted results, since their skillful selection of a particular winner could be either a big winner or a mild winner.

[INSERT FIGURE 3 ABOUT HERE]

[7] In order to construct this figure, we compute the Z-scores of stock returns in each month from December 31, 1988 to December 31, 2018. See Chincarini and Kim (2006) for the detailed methodology on computing stock Z-scores. We then rank stocks by highest Z-score in each period to lowest Z-score so that there are 500 ranked Z-scores. We then average over time each ranked Z-score with its corresponding Z-score in all other months. Of course, these will not necessarily be the same stock. We then plot the histogram of average Z-scores versus frequency and also fit a normal distribution curve to the data. Although we used Z-scores for simplicity, we could have also used stock returns. We could have just used one specific random month of data, rather than average over time. Neither of these alternative methods would change the nature of the results.

[8] The Z-scores are assigned according to the truncated distribution of positive Z-scores. Thus, if the portfolio manager had chosen 20 winners, then from the 250 winner Z-scores, we would assign Z-scores based upon their actual distribution, i.e. higher Z-scores would occur less frequently.

Constant Manager Skill

In the basic model of selectivity, Bolshakov and Chincarini (2019) assume that the manager's skill is constant regardless of how many stocks he picks. This assumption might be criticized since one might believe that the manager skill worsens over time as more stocks are picked.[9] One might also imagine a world where the manager believes he has skill, but is uncertain how much skill he has. That is, the skill might only be for a small handful of stocks and then his ideas run out. Another manager might worry even whether he has skill on any given pick. That is, maybe the manager believes he has skill on average, but there is an uncertainty with his skill in any given period or on any given stock.

In this section, we attempt to relax the assumption of constant manager skill regardless of how many stocks are picked and examine how the optimal selectivity changes. If the portfolio manager believes he has skill, then some of these results will be relevant. If the manager has no skill, then the manager should not be in the stock picking game and should be managing an index fund.

Steadily Declining Manager Skill

Traditional equity factor analysis, where the researcher uses historical data to periodically build theoretical portfolios based on some objective characteristic, can be thought of as an extension of these declining skill cases. For example, the portfolio returns from sorting stocks into deciles based on their cashflow/price ratio might produce an decrease in returns for every successive decile (i.e. cashflow/price getting smaller with each decile).

To simulate the effects of a decline in skill, the initial parameter of $\omega = 1.1$ was allowed to gradually decline as each stock was picked from the benchmark of stocks. Thus, prior to the first stock pick, $\omega = 1.1$ and linearly declines to $\omega = 1$ by the 500th stock pick. The declining ω leads to a straight decline in the expected return of the portfolio at each selectivity ratio, while the standard deviation of the portfolio declines in a convex fashion

[9]Worth noting is that even with a fixed skill parameter ω, the WNCH distribution exhibits a gradual decline in the probability of selecting winners. As more stocks are added into the portfolio, the number of winners left in the benchmark shrinks on average, and therefore the probability that the manager picks a winner for their next stock tends to shrink as well, all the way towards a 0% probability if all the winners have already been chosen.

versus selectivity. The resulting information ratio is optimal near the 50% level (see Figure 4), reminiscent of the bulk selection method rather than the sequential method. In Figure **??**, we also show the selectivity ratio and information ratio when skill does not decline (it's the line on the left most of the figure that extends beyond the top of the graph - Theoretical WNCH). What is interesting is that with a linearly declining skill of the form we suggested, the bulk selection method converges to the sequential selection method (i.e. Theoretical FNCH) at a 50% selectivity ratio.

[**INSERT FIGURE 4 ABOUT HERE**]

Jack Knife Skill

Another way to imagine skill not being constant is a portfolio manager that has good stock picking ability ($\omega = 1.1$) up to 30% of the universe and then no ability ($\omega = 1$) for any stock after that. For example, an analyst who has spent their entire career conducting long-only research may struggle with the concept of owning one-third of the universe. One reason that we use this so-called "jack-knife" scenario is to illustrate a more dramatic departure from the steadily declining skill situation. In that situation, although skill is declining, the manager always has skill. In the case of the "jack-knife", at a certain selectivity point, skill completely disappears. Selectivity theory would not apply in this case, since the manager would not select beyond the point at which his or her skill vanishes. For that reason the IR plot in Figure 5 reveals a clear break from the theoretical WNCH function and is maximized precisely at the drop off point.

[**INSERT FIGURE 5 ABOUT HERE**]

For a manager to know exactly how and when their skill declines seems unrealistic, but the two prior examples are helpful as they suggest a more general conclusion for sequential skill decay functions. It seems to be the case that the standard deviation of portfolio returns does not shift by much compared to the Base Case. Therefore as different functions are incorporated to reflect a more rapid skill decline, the optimal selectivity ratio will shift towards holding fewer stocks depending on how punitive that function is on the manager's expected returns. The maximum attainable IR will decline significantly though, so instead of shrinking their portfolio a skillful manager may be better served trying to hire like-minded analysts to their team, or devote time to educating new recruits in their successful methods.

Ability Saturation

Another way of adjusting the constant skill assumption is to assume that the manager believes to have skill only for a subset of the benchmark population. After that, the manager has no skill. In the real world such a scenario might reflect a manager's familiarity with certain sectors, or a quantitative model's inability to classify certain stocks such as initial public offerings or mergers that lack relevant financial data for the new company. Table 2 shows how the optimal selectivity changes, with the percentage of the benchmark that the manager has skill for, from 100% (i.e. constant ω) to 20% (the portfolio manager only has stock picking ability for 20% of the stock universe). In the simulation, we accomplish this, by allowing the manager to select stocks in a sequential manner, however, whenever the manager selects stocks, we cap the amount of winners at the saturation level.

[**INSERT TABLE 2 ABOUT HERE**]

Having fewer identifiable winners available to the manager lowers the information ratio that the manager can generate. However the course-correcting nature of the WNCH distribution is once again present, so the skillful manager is incentivized to hold more than 75% of the benchmark in all cases, in order to increase the probability that their skill shines through in the resulting portfolio. One can see from the table that in the original theoretical model, the optimal selectivity is 79%, but when the manager can only pick 20% of the winners in the universe, the optimal selectivity drops to 75%, but still far more than many managers hold.

Uncertainty in Manager Skill

Yet another way that manager skill might not be constant is that a manager may believe that on average he has skill, i.e. $\omega > 1$, however he is unsure about the exact value of his

ω. In each of our simulations, we choose the manager's ω from a distribution. In particular, $\omega \sim N(1.1, \sigma_\omega^2)$. We allow σ_ω^2 to vary from 0 to 0.20. The variability in a manager's skill could describe many real-life situations. For example, it could describe a manager that makes accurate predictions but the rest of the market fails to value stocks commensurate to his new information in a sufficient time period. It could also represent a period in which a manager's quantitative factor model or macroeconomic developments render the manager's skill ineffective within a particular time period. An example of such a changing period is 2018 and 2019, where momentum and value strategies have been lackluster.[10]

The simulation was done two ways. In the first method, every time the manager selects a stock, a random draw is made to determine the skill of the manager on the next draw. This is repeated throughout the selection. In the second method, we draw the ω parameter for a particular manager and keep it the same for the entire stock picking exercise. The results of these simulations are shown in Table 3. The first method might represent a portfolio manager that has skill on average, but for any given stock pick, or period of selection, has variability on his skill level. Sometimes the manager has bad days and sometimes good days, but on average is good. The second method could explain a manager that has bouts of good and bad in the different investment horizons. Over a career, this is surely possible.

[**INSERT TABLE 3 ABOUT HERE**]

The table shows that portfolio expected returns for a given selectivity ratio shrink at roughly the same rate for both methods. Overall, the methods have slightly different results. With method 1, the variability of ω has less of an effect on the information ratios than it does for method 2. With method 2, an increase in the uncertainty of ω causes a large decrease in the information ratio of the manager. However, the important point is that for both methods, the optimal selectivity is still around 80%. In the end, the actual value of the uncertainty doesn't matter. We could increase the uncertainty further, but it would simply increase the dispersion amongst different simulations, lead to a lower average information ratio, but the mean results of optimal selectivity would not change.

[10] It also could be a consequence of crowding in these factors (Chincarini (2012)).

A Simple and Practical Application of Declining Skill

Most practitioners are more concerned with practical, straightforward empirical relationships. In order to provide something along those lines, we use the Fama-French factor cash-flow-to-price as a quantitative factor to separate portfolios.[11] The factor portfolios are divided into deciles by their value of the cash-flow-to-price ratio every year and the monthly equal-weighted returns are computed. It has been shown that cash-flow-to-price can be considered a "value" factor in that companies with higher values tend to have higher average returns on average.

Traditional equity factor analysis, where the portfolio manager uses historical data to periodically build portfolios based on some objective characteristic, can be thought of as an extension of these declining skill cases. For example, the portfolio returns from sorting stocks into deciles based on their cashflow/price ratio produces an decrease in returns for every successive decile. While that pattern means the first decile has offered the best returns historically, a quantitative manager might still consider owning additional deciles to reduce tracking error against their defined universe. Moving from 10% to 20% selectivity would do just that, but it also introduces a deterioration of skill commensurate with the step down in returns. Further expansion of the selectivity ratio would lead to additional — but uneven — declines in both the numerator and denominator of the information ratio.

In Table 6 we produce the annualized returns of each decile from July 1951 through May 2019. One can also see the implicit selectivity ratios in that, each decile moving from one to ten is implicitly moving towards more and more stocks according to that particular selection criteria or that factor, i.e. cash-flow-to-price. The "best" stocks would be considered Decile 1, while the worst would be considered Decile 10. If you think of all the deciles as the benchmark portfolio, then, each additional decile is closer to the benchmark.

[**INSERT TABLE 6 ABOUT HERE**]

In Table 6 we also show the information ratio historically of different selectivity ratios. In this particular case, the 40% selectivity obtained the highest historical information ratio.

[11] Source: https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

One way to view these results is to consider the real world of factor selection as a combination of linearly declining skill and jack-knife skill as we presented earlier. In this case, the results might naturally lie at 40% selectivity.

In a more general sense, the cash flow yield factor also represents one factor over one sequence of history.[12] In practice, quantitative portfolio managers don't buy all deciles. They might buy a portion of the first decile and wait a period, then buy or sell depending on the movements of stocks according to that factor. Thus, their behavior is more like a dynamic use of a factor that a one-period investment choice.[13]

Equal-Weighted Benchmark

The fourth assumption in Bolshakov and Chincarini (2019) is that the benchmark is an equal-weighted benchmark. This is not a strong assumption, in the sense, that many benchmarks are equally-weighted. In this section, we consider how the results would change if we use a market-capitalization weighted benchmark like the S&P 500. In practice there is considerable size variation to companies within a benchmark, particularly for large-cap benchmarks where the largest stock can be hundreds of times the size of the smallest stock. Many popular benchmarks such as the S&P 500 weight their holdings according to their market capitalizations. That methodology can lead to a handful of stocks having an outsized influence on the benchmark's return. Figure 6 shows the returns of S&P 500 stocks for the month of January 2019. The stocks are listed in order of smallest return to largest return and each company's market capitalization is depicted as a circle proportional to its market capitalization. The underperformance of a handful of very large stocks such as Microsoft,

[12]While an optimal selectivity ratio can be found in examples like the one above, its precise location will be quite sensitive to the historical data sample. Paul Samuelson used to say that even the U.S. stock market was just one sample of history. He used to suggest a thought experiment: Write down stock returns for every minute or every hour or every day for the past 200 years on pieces of paper, put them into a big bowl or hat and mix thoroughly. "Now lets draw out a new history of 200 years," he continued. "And lets do that so we have 20,000 different histories." He noted that in most of these alternate histories, the probability increases that you'll be ahead the longer you invest. "But with the same remorseless certainty," he explained, "there will be some long periods when you will lose, and the longer the horizon the greater the amount of loss. But of course you don't see these alternative runs of history." Updegrave (2009). It might also be of interest to build a mathematical model that translates the empirical ability of a factor into the setting of selectivity theory as a probability of selecting a good stock.

[13]This idea is discussed later in the paper.

Apple, and Alphabet pulled the market cap-weighted average return down below the equal-weighted average return. The return for the actual S&P 500 index that month was 8.01%, whereas an equal-weighted version of the same index returned 9.85%.

[INSERT FIGURE 6 ABOUT HERE]

In order to relax the assumption of an equal-weighted benchmark, we created simulations of selectivity, but with market capitalization weighted portfolios and benchmarks using S&P 500 stocks. The first question becomes what is an appropriate benchmark for the market capitalization weighted portfolio. Given that half the stocks are winners and half the stocks are losers, how do we allocate these in our benchmark portfolio? One idea would be to allocate the winners and losers such that the weighted average return is close to zero. That is, the return of the market capitalization weighted benchmark will be given by:

$$r_{MC}^{BM} = \sum_{i=1}^{N} w_i r_i = \sum_{i=1}^{N} \frac{MC_i}{\sum_{i=1}^{N} MC_i} r_i \tag{1}$$

where MC_i is the market capitalization of company i and r_i is the return of stock i. Since we wish to keep the binary nature of the returns the same as the base case, we will have 50% of the stocks with a 10% return (or any other return for that matter) and 50% of the stocks with a -10% return. Thus, we can write the return of the market capitalization benchmark as

$$\begin{aligned} r_{MC}^{BM} &= \sum_{k=1}^{n_g} \frac{MC_k}{\sum_{i=1}^{N} MC_i} r_g + \sum_{j=1}^{n_b} \frac{MC_j}{\sum_{i=1}^{N} MC_i} r_b \\ &= \frac{1}{MC} r_g \left(\sum_{k=1}^{n_g} MC_k - \sum_{j=1}^{n_b} MC_j \right) \end{aligned} \tag{2}$$

where MC is the total market capitalization of all the stocks. Thus, the return of the benchmark in this one period setting will be the difference in the total market capitalization of the good stocks minus the total market capitalization of the bad stocks multiplied by some factor. One choice for a benchmark would be one that, similar to our equal-weight benchmark, produces an average return of zero. This would be one with 50% of the market capitalization of companies as good stocks and 50% as bad stocks. There are many ways this could be chosen, so long as we meet this criterion. We could also choose another benchmark where all of the winner stocks are the largest companies in the benchmark. This would have an average portfolio return that is positive. We could also choose a benchmark where all of the loser stocks are the largest companies in the benchmark. There is no correct benchmark. Ultimately we must choose a market cap benchmark where the good and bad returns are distributed amongst stocks in the benchmark.

For our analysis, we created five benchmarks. The zero return benchmark we call "Neutral Size", the benchmark where the largest 250 stocks are winners, we call "Strong Large-Cap Tilt", the benchmark where the largest 250 stocks are losers, we call "Strong Small-Cap Tilt", and then we create two portfolios that are in-between "Strong Large-Cap Tilt" and "Neutral Size" and "Strong Small-Cap " and "Neutral Size".[14] These are "Large-Cap Tilt" and "Small-Cap Tilt". We believe this gives us a wide variety of market cap benchmarks from winners extremely tilted to big stocks and to winners extremely tilted towards smaller companies.

Once the benchmark was selected, the procedure for generating portfolios was the same. First, we drew from the Wallenius to determine whether we had a winner stock or a loser stock. Then we randomly assigned that winner or loser return to a company, then when we finished drawing the number of stocks for a particular selectivity ratio, we computed the market capitalization weighted and equal-weighted return of every portfolio. We then repeated this procedure for the M simulations for every selectivity ratio. The results of the simulations are contained in Table 5.

[INSERT TABLE 5 ABOUT HERE]

[14]In order to create a zero return benchmark, one can use optimization tools or other methods, but there will be a large number of solutions to this problem. In order to construct our benchmark, we simply took a random draw of market cap and assigned the stock a winner return, we then took another draw from the benchmark list without replacement and assigned a winner return, and so on and so forth until we had a return for all N benchmark stocks. At every draw, we also computed the sum of winner market capitalizations and loser market capitalizations (i.e. $\sum MC_k$ and $\sum MC_j$ through the nth selection of stocks). If the sum of winners is greater than the sum of losers, we assign the next winner return to a smaller company (i.e. a company in the bottom 250 of companies by market capitalization). Once again, this is randomly picked. Again, this is only one of many ways we could have assigned stocks to create a zero return benchmark.

The table shows that if the benchmark is market capitalization weighted and the best performing stocks are large-cap stocks, then optimal selectivity level will be far from what the theory predicts and the information ratio of the portfolio manager will be poor relative to the benchmark *for equal-weighted portfolios*, which is to be expected. This is reversed if the best performing stocks are small-cap stocks. This is quite natural due to the fact that equal-weighting places less emphasis on the returns of stocks with higher market capitalizations. However, if the portfolio manager with skill weights his or her portfolio using a market capitalization weighting, then the results are closer to predictions of selectivity theory. That is, whether the best performing stocks are large-cap ("Strong Large-Cap Tilt") or small-cap ("Strong Small-Cap Tilt"), the average optimal selectivity ratio is between 70 and 83 percent and all of the information ratios are positive.

In summary, the least challenging assumption in the theory of selectivity is the benchmark weighting. However, we have shown that if the benchmark is a market capitalization weighted benchmark, then it would make sense for the portfolios to be market capitalization weighted as well to achieve the highest information ratio. The optimal selectivity ratios would also be slightly higher in the 70% to 83% range.

Dynamic Selectivity

Until this point, the optimal selectivity discussion has considered a one-period horizon. A portfolio manager chooses a basket of stocks to own from a benchmark and holds that portfolio. It could be that the portfolio manager simply repeats the exercise at the next rebalancing date. However, this is not typically the way portfolio managers behave. A more likely scenario is that a portfolio manager has a method to pick stocks, whether it's fundamental or qualitative or a more automated quantitative process. For the former, as new ideas come in, the manager might purchase a new stock or group of stocks from time to time. For the latter, as new information becomes available, the portfolio manager may process his quantitative stock return model and decide to purchase new stocks. Other factors, such as tax issues and transaction costs, may also cause a manager to repeatedly adjust the holdings in his portfolio rather than buy all stocks at the same time.

In this part of the paper, we study the way in which a more realistic way of adding stocks to one's portfolio behaves in relation to selectivity theory.

Portfolio Formation Methodology

Similar to the simulation process in other parts of this paper, portfolios are simulated by drawing stocks without replacement from a benchmark. For each individual simulation i, a portfolio is created by assuming some fixed number of stocks j are added to the portfolio each month. Each individual addition will eventually exit the portfolio based on an investment horizon chosen at random. These investment horizon draws are made independent of all other aspects of the simulation. The investment horizon probability distribution can take on any form, however for the results presented in this paper, we choose the following criteria: 1. The minimum holding period for any stock purchase was three months; 2. The maximum holding period for any stock was 48 months or 4 years; 3. The average holding period for any stock was 12 months (and for 24 months for another simulation); and 4. The selection of stock holding period was obtained from a probability distribution that was smooth and declining for longer holding periods.

The probability distribution function that we chose for the holding period comes from the following function:

$$f(n) = \frac{1}{48(n-2)^x} \; \forall n \in 3, 4, ..., 48 \tag{3}$$

where x is a parameter to be chosen. One can see that this functional form leads to a much lower value of $f(n)$ when n is larger versus smaller, reflecting the idea that the manager is more likely to pick stocks with a shorter investment horizon. In order to make use of this as a probability distribution, we had to normalize the value such that the sum of discrete outcomes equals one. Thus, the probability of obtaining a particular holding period, n, is given by:

$$\text{Prob}(n) = \frac{f(n)}{\sum_{i=3}^{48} f(n)} \tag{4}$$

Finally, in order to establish a mean holding period of 12 months, we needed a value for the parameter x such that the mean of the holding period distribution would be twelve (i.e. $\sum_{i=3}^{48} f(n) = 12$). The value of x that achieves this desired property is $x = 1.0325$.

Figure 7 shows the effective distribution function of the holding periods from which we draw for each stock selection by the manager. Besides the mean holding period of twelve months, some other characteristics are worth noting. More than 50% of the stock selections will have a holding period of less than six months. Only 15% of the stocks will have a holding period of longer than two years and only about 6% will have a holding period of more than three years. Generally speaking, our holding period distribution is very reflective of a portfolio manager that generally holds stocks less than one year. This holding period is realistic. If one looks at the turnover of the S&P 500, it is about 100 days or about three months. The reason for our choice of a minimum holding period of three months is that in coincides with quarterly reports from companies and earnings calls, which would be a natural updating point for quantitative models, as well as for fundamental analysis.[15]

[**INSERT FIGURE 7 ABOUT HERE**]

Before every stock selection, we randomly drew from the holding period distribution. Then we determined whether the pick was a success or not, based on the Wallenius distribution. As we moved forward in the simulation, when a stock's "time had come", meaning it's holding period was over, the stock will be dropped from the portfolio and a new stock will be chosen to replace it. In addition to the removal of a stock at the end of its holding period, we also removed stocks in the S&P 500 that were delisted, went bankrupt, were acquired, or had other corporate actions occur. When these events occured, we attempted to remove the stock in the previous month at the best estimated selling price of the company at that time in history. All stocks in the portfolio at any given time were equally-weighted. Our benchmark was the equally-weighted S&P 500.[16]

[15] We also present the results for a 24-month average holding period which makes the distribution close to uniform.

[16] The equally-weighted S&P 500 index was actually created by one of the co-authors in 2003 and now trades as an ETF with the ticker symbol, RSP. For more information, see http://ludwigbc.com/pubs/SPEWIWhitePaper010703.pdf.

The sequential and dynamic stock selection was done as follows. In any given month, t, the portfolio contained a subset of stocks from the S&P 500. The portfolio manager will choose a new stock from stocks in the S&P 500 that were not already in the portfolio and did not have missing data for the following month. In order to appropriately assign winner and loser stocks to the portfolio, once the holding period was known of the particular stock that the manager chose, we separated all of the eligible stocks by their *actual return* over the holding period into winners and losers, which means a compound return above or below that of the benchmark.[17] It was at this point that we used the Wallenius distribution and allowed the manager to pick a stock with the parameter of $\omega = 1.1$. The process was then repeated, where the portfolio manager draws a holding period, then draws a stock which could be a winner or a loser.

As already mentioned, our sample S&P 500 data is from December 31, 1988 to December 31, 2018. However, for the dynamic simulation we used the period of data between December 31, 1992 and December 31, 2014. The first four years were used to build up the portfolio to steady-state size and the last four years were excluded due to lack of forward return information.[18] Once we arrived at the steady-state, the portfolio of the manager would have roughly n_t stocks in the portfolio, which divided by the number of stocks in the benchmark, N, equalled to the selectivity ratio. In steady-state this will be equal to the average holding period of the stocks multiplied by the number of stocks that the portfolio manager adds per month divided by the number of stocks in the benchmark (Little (1961)). That is,

$$\bar{S}_t = \frac{\bar{n}_t}{N} = \frac{j\bar{h}}{N} \tag{5}$$

where $\bar{S}_t$ is the average selectivity ratio, $\bar{n}$ is the average holdings of the portfolio, j is the number of stocks that are added to the portfolio by the portfolio manager each month, and

[17] For example, if the portfolio has only 1 stock, there will be 499 stocks available to trade. Suppose the randomly drawn investment horizon is six months, then we will compute the forward-looking total returns for every one of the 499 stocks and call those that beat the benchmark winners and those that do not losers. If there is any missing return data for these stocks in any month of the holding period, we assign the return as 0%.

[18] In order to get to a steady-state, we found that the number of months that must elapse are equal to the maximum holding period of any stock.

$\bar{h}$ is the average holding period of each stock in the portfolio, which in our case, equalled twelve.

Thus, in order to modify our simulation for different selectivity levels, we could either change the number of stocks the manager picked per month, j, or the mean holding period of the manager, $\bar{h}$. We did this by altering the number of stocks that the manager chose per month. In order to compute the information ratio for the portfolio manager, we computed the monthly portfolio return based on the actual return of the stocks in the portfolio and the benchmark return.[19] The portfolio was rebalanced every month to be equally-weighted amongst all the stocks.

Dynamic Simulation Results

Before jumping to the results of the simulations, we first depict how a sample portfolio looks over our sample period. Figure 8 shows one of our simulations, where $j = 25$, $\bar{h} = 12$, the steady-state average number of stocks is 300, and hence the average selectivity ratio is 60 percent.[20]

The figure also shows the four years that we exclude from the beginning of the analysis and at the end of the analysis. From January 1992 to December 2014, the portfolio fluctuates around the mean 300 stocks and/or 60 percent selectivity ratio.

[INSERT FIGURE 8 ABOUT HERE]

[INSERT TABLE 7 ABOUT HERE]

Figure 9 plots the average information ratios for each selectivity ratio along with the polynomial function fit through them. For each selectivity ratio, we conducted 10,000 simulations and each simulation resulted in a different information ratio. Figure 10 graphs the distribution of information ratios for each selectivity ratio. The simulated maximum is technically achieved at 33 new additions per month, which using Eq. (5) corresponds to a

[19]The benchmark was the equally-weighted S&P 500 rebalanced monthly.

[20]The dynamic simulation is computationally expensive. We used the R programming language "parallel" package to construct our 10,000 simulations for each selectivity ratio. We reduced the number of simulations from 100,000 to 10,000 due to the computational time required to complete the simulations.

selectivity ratio of 79.2%, just as the theoretical, one-period Wallenius distribution would recommend. Panel A of Table 7 displays IR metrics for the 79.2% case as well as other selectivity ratios. Panel B summarizes the range of IR results within each of those selectivity ratios, and Panel C provides volatility metrics, again looking at the cross-section of results within each selectivity ratio.

The impact that different skill level and holding period assumptions have on the time series simulation framework is presented in Table 8. Each column summarizes the results at the optimal selectivity ratio for a separate round of simulations across selectivity ratios. The left three columns increase the skill parameter while holding period assumptions remain unchanged, while the right three columns lengthen the average holding period while leaving the skill parameter unchanged. Panel A summarizes results for the sequential selection approach while Panel B shows results for a bulk-selection approach at each month in the simulation.[21]

In all cases the simulated distributions fall within the theoretical range of 50% to 79%. At one month holding periods the optimal ratios reach those endpoints exactly, as these cases are essentially monthly replications of the one-period simulation tests conducted earlier. For both the sequential and bulk selection frameworks, as skill increases and as the average holding period decreases the level of IR achieved at the optimal selectivity ratio increases. In all sequential selection scenarios the optimal selectivity ratio remains fixed at the theoretical endpoint, just under 80%.[22] For bulk selection cases, the optimal selectivity ratios are a few percentage points above the theoretical value of 50%. This shift towards the Wallenius side of the spectrum reveals that there are some course-correcting properties to resampling from the same benchmark over time. Even though each monthly sampling exercise has no memory between draws, once a given winner has been selected into the portfolio, it cannot be bought again. Thus, if a manager gets excessively lucky for a few months in a row then

[21] To conduct the bulk selection simulations, the process was changed such that approximately all monthly j picks were made simultaneously and for the same average holding period, h, using the computational logic presented in Appendix Appendix A.

[22] For the case of 24 month average holding periods the optimal selectivity ratio of 76.8% is actually the largest ratio below 80%, there are just fewer selectivity ratio points to choose from as the number of new adds each month shrinks in order to accommodate the longer holding periods.

their future binomial draw processes will be excessively tilted towards picking losers, similar to how the Wallenius distribution recalculates selection probabilities after each intra-month selection. In practice, the actual way stocks are selected into the portfolio will fall somewhere in between bulk and sequential selection, which will result in optimal selectivity somewhere between 50 and 80 percent. However, as Table 8 clearly demonstrates, shifting the portfolio selection process more towards sequential selection method increases both, the information ratio and optimal selectivity.

[**INSERT FIGURE 9 ABOUT HERE**]

[**INSERT FIGURE 10 ABOUT HERE**]

[**INSERT TABLE 8 ABOUT HERE**]

Conclusion

The work in this paper attempts to further test and examine assumptions about the optimal selectivity in an enhanced portfolio (Bolshakov and Chincarini (2019)). The specific purpose of this paper was to relax assumptions about the investment environment using simulations to investigate the impact of the assumptions. When we relax the assumptions in the original theoretical paper, we find the general implications of the theory still hold. However, when there are certain limits to skill, the optimal selectivity differs from the general theory as would be expected. In other words, we find that even with the relaxed assumptions, it makes sense for an enhanced index manager to hold somewhere between 50% and 80% of the benchmark provided the manager has skill. If the manager has no skill, then the best course of action, is obviously, to fully index. If the manager has varying degrees of declining skill, i.e. a limited ability to pick stocks, then the optimal point will be achieved at a lower selectivity ratio.

We also examined the impact of dynamic portfolio management on the optimal selectivity ratio. That is, rather than a portfolio manager choosing a group of stocks in a one horizon setting, we examine a portfolio manager that selects new stocks every rebalancing period and selects stocks with different holding periods. We find that as long as the portfolio manager has skill the optimal selectivity is somewhere between 50% and 80%.

This research is promising for the theory of selectivity. However, there are important areas for future research. First, it might be possible to use these techniques to identify skillful managers from a dataset of professional manager returns and selectivity ratios. Second, it might be enlightening to examine actual historical return data in combination with some objective skill-like criterion for selecting stocks to determine whether optimal selectivity ratios are confirmed with very practical implementation methods.

In addition to the promising application for investing, we also believe that these ideas provide an alternative to potential harming effects of total passive management or crowding in passive asset management. As of the writing of this paper, 26% of the S&P 500 is controlled by passive investment organizations. The frustration with poorly performing portfolio managers has led to this dramatic shift to passive investing, but perhaps there is room for a more optimal investing style that selects a large portion of the benchmark, yet still cares deeply about performance and high information ratios.

Clearly, this interesting new theory needs to be examined and scrutinized more thoroughly. However, it is comforting that as a first step, these simulations confirm some of the original findings of the theory when the assumptions are relaxed. Another very interesting aspect of selectivity theory is the conclusion that even an active manager with superior skill might want to invest in the majority of stocks in the underlying index.

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Appendix A Fisher Simulation Methodology

In terms of simulations, the Fisher distribution, i.e. where managers pick sequentially, is much more complicated than the Wallenius distribution, i.e. where managers pick in bulk. The reason is quite obvious. If we decide on a selectivity ratio, then we can use the Wallenius to select the good and bad stocks for the bulk selection method. However, when we select sequentially and we wish to have different selectivity ratios, we must consider the path of picks prior to the particular selectivity ratio. This makes the simulation more difficult. In this appendix, we describe the steps that we implemented to simulate the sequential selection method, as well as present the results for the relaxation of assumptions of Bolshakov and Chincarini (2019) that were presented for the Wallenius distribution in the main body of this paper.

Description of Simulation Steps

Step 1: Construct Appropriate Benchmark

Similar to the Wallenius Monte Carlo simulations, a theoretical benchmark is constructed from which the manager will randomly draw stocks. The benchmark can be as simple as the original Base Case, where 250 winning stocks return +10%, 250 losing stocks return 10%, and all constituents are equal-weighted. Or alternatively, the benchmark can be tailored to different winner/loser populations, more complex return distributions, or market cap weighted schematics.

Step 2: Calculate Probabilities for all Selectivity Ratios

One of the primary distinctions between the Fisher and Wallenius distributions is that in the former the number of stocks selected (and therefore the selectivity ratio) is not known in advance; the total number of stocks held is itself a random outcome from independent binomial processes. If one attempts to plug in the Base Case WNCH probabilities for $\omega = 1.1$ to these binomial processes (i.e. 0.523 for winners and 0.476 for losers), then the resulting FNCH simulated portfolios will be tightly clustered around a 50% selectivity ratio. It would take a massive number of simulation trials at least in the trillions but likely much larger than that to generate simulated observations of portfolio sizes far away from 50% selectivity.

However just as a quantitative manager can tailor their model to be more or less strict in its rules, so can the binomial probabilities assigned to the winner and loser populations be shifted to focus the FNCH simulations around a desired selectivity ratio. Because the binomial distributions are independent, the expected number of stocks selected from each subpopulation will be additive (see (8) below). In the algebra that follows X_1 and X_2 are the number of winners and losers randomly drawn, m_1 and m_2 represent the number of winners and losers in the benchmark, and p_1 and p_2 signify the binomial probability of drawing a winner or drawing a loser, respectively, from those subpopulations.

$$E[X_1] + E[X_2] = m_1 p_1 + m_2 p_2 \tag{6}$$

If one is targeting a given selectivity ratio S, then the binomial expected value in Equation (6) can be rewritten as:

$$S = \frac{m_1 p_1 + m_2 p_2}{m_1 + m_2} \tag{7}$$

For an assigned skill level ω there is a direct relationship between p_1 and p_2. Since each of the two subpopulations is following a binomial process of either (a) select a winner vs. select nothing, from the subpopulation of winners; or (b) select a loser vs. select nothing, from the subpopulation of losers, the skill parameter can be expressed in the following ratio:

$$\omega = \frac{\frac{p_1}{1-p_1}}{\frac{p_2}{1-p_2}} \tag{8}$$

The ratio in Equation (8) can be reorganized in terms of p_2:

$$\begin{aligned}
\omega &= \frac{p_1 - p_1 p_2}{p_2 - p_1 p_2} \\
\omega p_2 - \omega p_1 p_2 + p_1 p_2 &= p_1 \\
p_2(\omega - \omega p_1 + p_1) &= p_1 \\
p_2 &= \frac{p_1}{(\omega - \omega p_1 + p_1)} \qquad (9)
\end{aligned}$$

At this point p_2 can be substituted into equation Equation (9) to solve for p_1. Since the subpopulations m_1 and m_2 add up to the benchmark population N, the notation can be simplified along the way. The equation can be re-organized into a traditional quadratic expression set equal to zero.

$$\begin{aligned}
S(m_1 + m_2) &= m_1 p_1 + \frac{m_2 p_1}{(\omega - \omega p_1 + p_1)} \\
SN - m_1 p_1 &= \frac{m_2 p_1}{(\omega - \omega p_1 + p_1)} \\
SN\omega - SN\omega p_1 + SNp_1 - \omega m_1 p_1 + \omega m_1 p_1^2 - m_1 p_1^2 &= m_2 p_1 \\
(\omega m_1 - m_1)p_1^2 + (SN - SN\omega - \omega m_1 - m_2)p_1 + SN\omega &= 0
\end{aligned}$$

The standard solution to this equation is

$$\begin{aligned}
p_1 &= \frac{-(SN - SN\omega - \omega m_1 - m_2) \pm \sqrt{(SN - SN\omega - \omega m_1 - m_2)^2 - 4(\omega m_1 - m_1)SN\omega}}{2(\omega m_1 - m_1)} \\
p_1 &= \frac{SN(\omega - 1) + \omega m_1 + m_2 \pm \sqrt{[SN(\omega - 1) + \omega m_1 + m_2]^2 - 4SN\omega m_1(\omega - 1)}}{2m_1(\omega - 1)} \qquad (10)
\end{aligned}$$

Being a probability value p_1 cannot exceed 1, and for a manager to have skill ω must be greater than 1. Together these constraints can be used to exclude the addition possibility in Equation (10) . In the numerator both the left and right hand side of the $\pm$ will always be greater than ωm_1. The right hand side will always be smaller than the left due to its subtraction of $4SN\omega m_1(\omega - 1)$. As for the denominator, its $\omega - 1$ term restricts it to be less than $2\omega m_1$. Therefore the only way to maintain $0 \leq p_1 \leq 1$ is to subtract the right hand side of the numerator from the left.

Now with a single, solvable function, a complete Fisher distribution can be simulated by looping through all possible S values to create a series of p_1 and p_2 probability pairs to use, one pairing for each distinct selectivity ratio.

Step 3: Conduct Independent Binomial Draws in each Subpopulation

With the probability pairs computed, random binomial draws are run independently on the winner and loser subpopulations to determine how many winners and losers are selected into the portfolio. The expected selectivity ratio will remain true to the targeted levels found in Step 2, but each sampled portfolio size may deviate from that size. Nevertheless, all simulations are grouped according to their targeted selectivity ratio rather than their realized selectivity ratio. Since the latter cannot be known in advance by the manager, it would be meaningless to draw conclusions from the ex-post size results.

Within each subpopulation it is assumed that each stock is equally likely to be selected. For example, if the binomial draws indicate 25 winners and 20 losers are to be chosen, then 25 winners are drawn from the winner subpopulation according to a uniform random distribution, and the losers are drawn from their subpopulation the same way. These selections are made without replacement, which allows for the real-world complexities to be introduced, as each stock within the benchmark can be tagged with its unique market cap or return characteristics as desired.

Step 4: Calculate Portfolio Returns

Lastly, portfolio returns for a given simulation are calculated based on the stocks identified from the benchmark and their pre-assigned returns. Holdings are assumed to be equally-weighted unless noted otherwise. Steps 1 to 4 are then repeated within each targeted selectivity ratio according to how many simulations are specified. Following completion of all simulations, cross-sectional performance metrics are calculated within each targeted selectivity ratio, and then a polynomial function is fit across those cross-sectional results to accurately locate the optimal selectivity ratio.

B2. Base Case Monte Carlo Simulations

The same assumptions used in the WNCH Base Case are applied to the above Fisher simulation structure, with 100,000 simulations conducted at every possible targeted selectivity ratio between 10% and 90%. The polynomial-fitted line from these simulations is then compared to the theoretical FNCH Expected Returns and information ratios as proposed by (Bolshakov & Chincarini (2019)). Across all selectivity ratios the two approaches produce IRs that have a correlation of 0.9985. This proximity indicates that the above simulation structure is appropriate for modeling the FNCH distribution and studying the effects of real-world complexities.

Figure 11 plots the resulting information ratios from these simulations, alongside the theoretical IRs.

[INSERT FIGURE 11 ABOUT HERE]

Relax Assumption 1: More Losers than Winners

Figure **??** shows the selectivity ratio and information ratio in the case of 40% winners alongside the polynomial function fit through simulations of the same population. With fewer opportunities for their skill to shine, the manager's optimized IR declines; however their optimal behavior does not. Similar to the findings for WNCH simulations, the optimal selectivity ratio does not deviate materially from the Base Case (which for Fisher is 50%).

[INSERT FIGURE 12 ABOUT HERE]

B4. Relax Assumption 2: Empirical Distribution of Returns

Substituting the empirical distribution of returns in for the binary ±10% assumption does not alter the optimal selectivity ratio either. Using the S&P 500 z-scores described in the WNCH simulations produces a similar parabola as seen in the Base Case Fisher distribution, although the IR for each selectivity ratio shifts down because of the greater variability within the benchmark's returns. Figure 12 plots these results.

[INSERT FIGURE 13 ABOUT HERE]

Relax Assumption 3: Decaying Manager Skill

Most of the skill decay scenarios explored for the Wallenius distribution cannot be tested under Fisher conditions as the draws must happen simultaneously to create the Fisher distribution. The only case that realistically can be simulated is for the skill level to vary across portfolio simulations, mimicking a manager facing different regimes that periodically favor or punish their long-term skill. While the addition of extra variability would lower the manager's optimized IR, the manager would continue making their construction decision the same way as in the Base Case, just using $E(\omega)$ instead of ω as their assumed skill parameter. As long as $E(\omega) > 1$, that construction decision will remain optimized at a selectivity ratio of 50%.

Relax Assumption 4: Cap-Weighted Returns

Although we do not show the results, building a market cap-weighted benchmark also leads to similar conclusions as those found studying the WNCH distribution. The standard deviation of returns for the manager's portfolio expands, which lowers their potential IR. If the market turns out to be narrow the manager will have an ex-post desire to have weighted their portfolio according to market caps; conversely if the market turns out to be wide (i.e. smaller stocks outperform) the manager will wish they had equal-weighted their holdings.

On an ex-ante basis though, across the different regimes that can materialize, the manager maximizes their IR by weighting their holdings proportional to their market caps, and targeting a selectivity ratio slightly below 50%.

Appendix B Main Body Figures

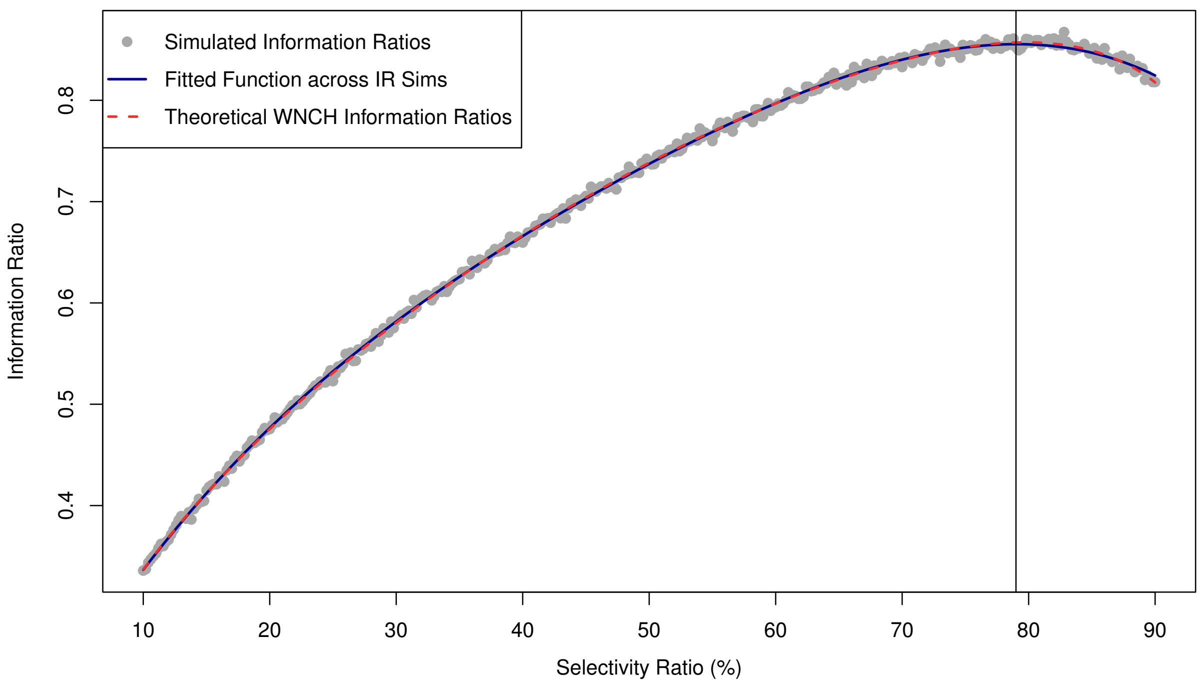

Figure 1: **Baseline Simulations.** This figure shows the baseline simulations of each selectivity ratio and its corresponding information ratio. The dots represent the average information ratio from 10,000 simulated portfolios at that particular selectivity ratio. Also shown is a fitted line across the simulation averages and the threoetical Wallenius information ratios for each selectivity ratio. The optimal selectivity ratio is at 79%. The parameters for the simulations are $N = 500$, $\omega = 1.1$, $r_g = 0.10$, $r_b = -0.10$, and $r_{BM} = 0$.

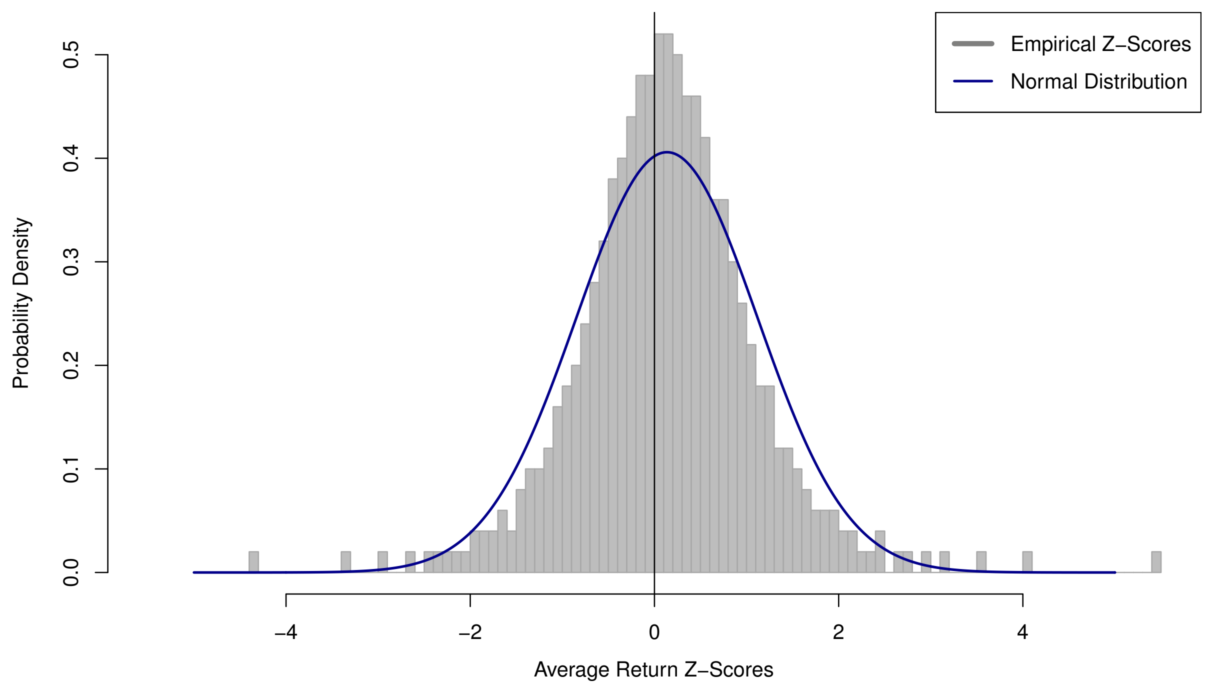

Figure 2: **Average Cross-Sectional Z-scores for S&P 500 Stock Returns.** This figure shows the distribution of Z-scores of stock returns for stocks in the S&P 500 from December 1988 to December 2018. In each period, cross-sectional Z-scores are created based on the returns of each stock in the S&P 500. This is repeated for every month in the sample period and the average Z-scores compute along with the frequency of occurrence. A normal distribution is also fitted to this histogram

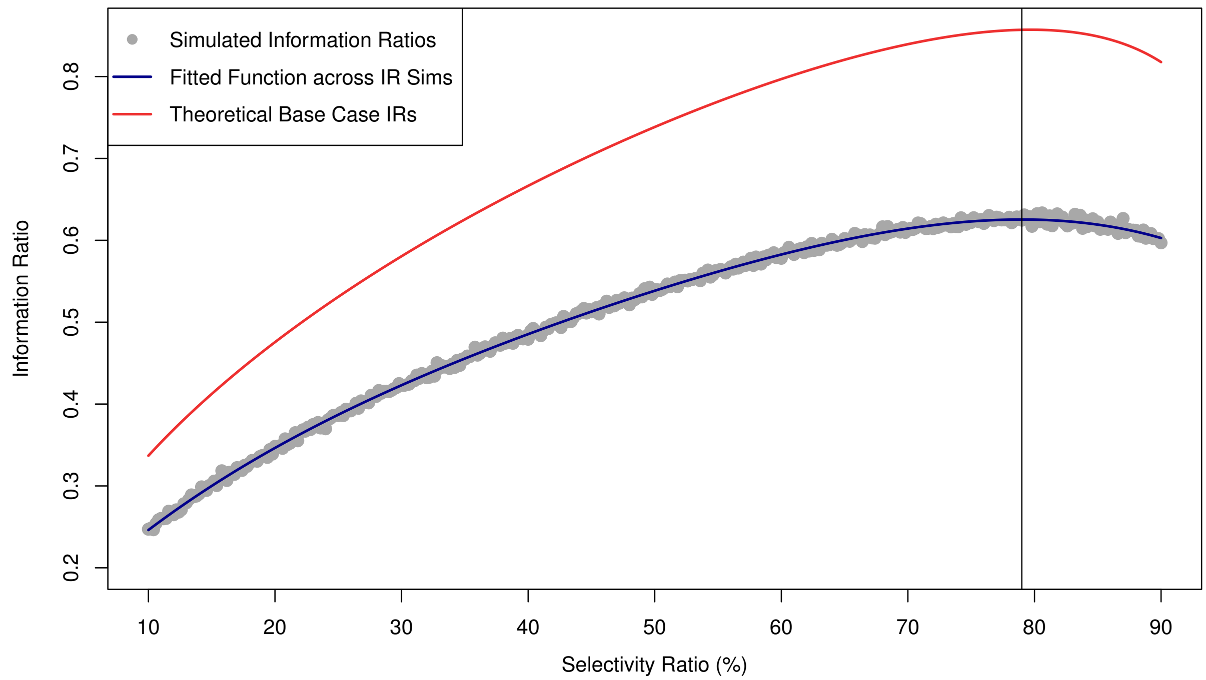

Figure 3: **Selectivity Ratio and information ratio for Empirical S&P 500 Returns.** This figure shows each selectivity ratio and its corresponding information ratio when returns are not binary for good and bad stocks, but are based on the cross-sectional Z-scores of actual S&P 500 individual stock returns. The dots represent the average information ratio from 10,000 simulated portfolios at that particular selectivity ratio. Also shown is a fitted line across the simulation averages and the threoetical Wallenius information ratios for each selectivity ratio. The parameters for the simulations are $N = 500$, $\omega = 1.1$, $r_i = \bar{z}_i$, and $r_{BM} = 0$.

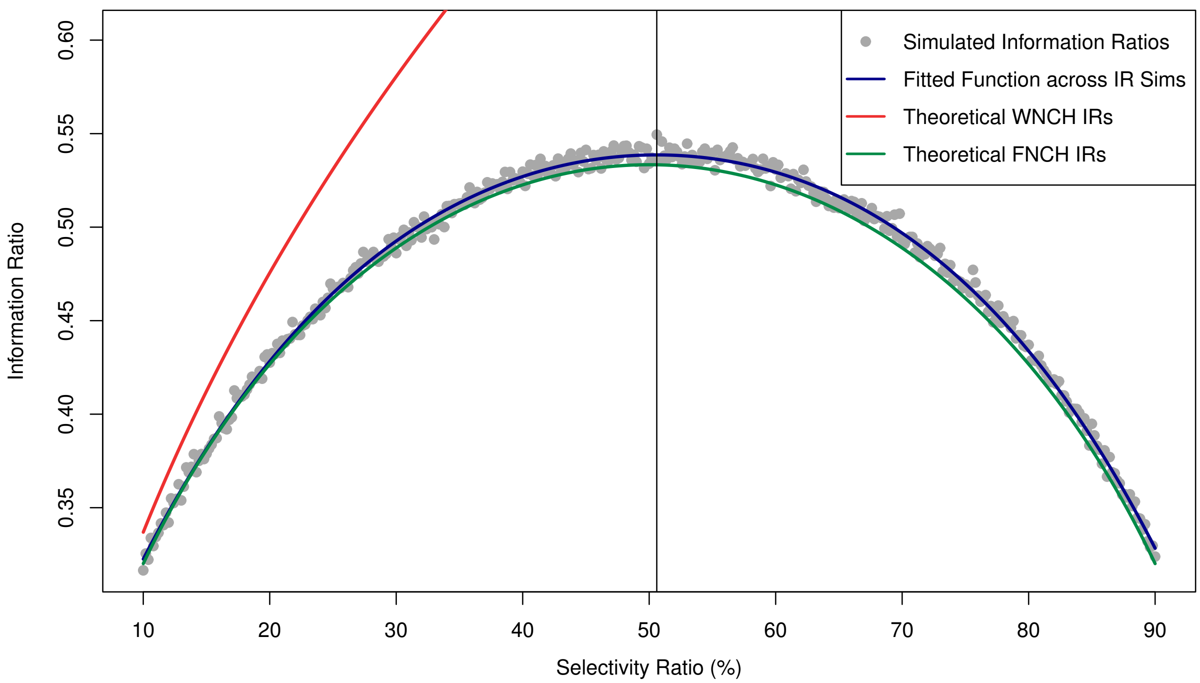

Figure 4: **Selectivity with Steadily Declining Manager Skill.** This figure shows the information ratios and selectivity levels when the manager's skill is steadily declining from $\omega = 1.1$ on the first stock pick to $\omega = 1$ by the last stock pick. The dots represent the average information ratio from 10,000 simulated portfolios at that particular selectivity ratio. Also shown is a fitted line across the simulation averages and the threoetical Wallenius information ratios for each selectivity ratio. The optimal selectivity ratio is at 50%. The parameters for the simulations are $N = 500$, $r_g = 0.10$, $r_b = -0.10$, and $r_{BM} = 0$.

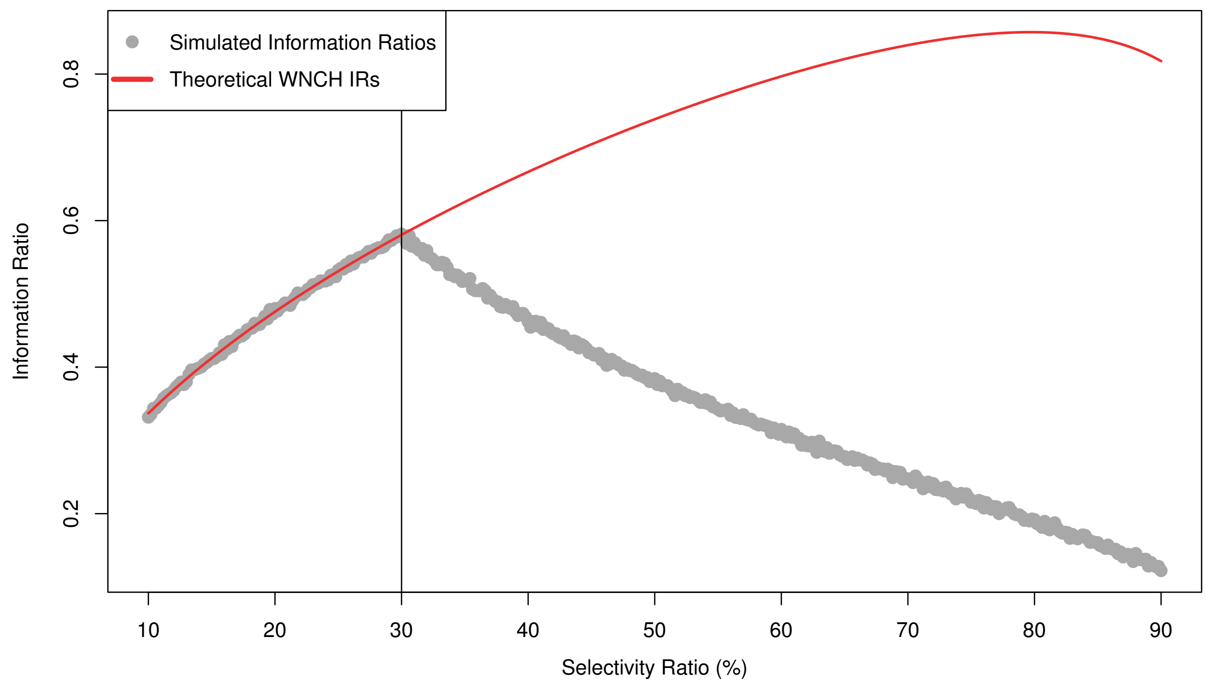

Figure 5: **Selectivity with Jack Knife Skill.** This figure shows each selectivity ratio and its corresponding information ratio when the portfolio manager's skill completely vanishes at a certain selectivity ratio. That is, the manager has skill until a certain selectivity and then it reverts to random. The dots represent the average information ratio from 10,000 simulated portfolios at that particular selectivity ratio. Also shown is a fitted line across the simulation averages and the threoetical Wallenius information ratios for each selectivity ratio. The parameters for the simulations are $N = 500$, $\omega = 1.1$ for $\phi = n/N < 0.30$, $\omega = 1$ for $\phi = n/N \geq 0.30$, $r_g = 0.10$, $r_b = -0.10$, and $r_{BM} = 0$.

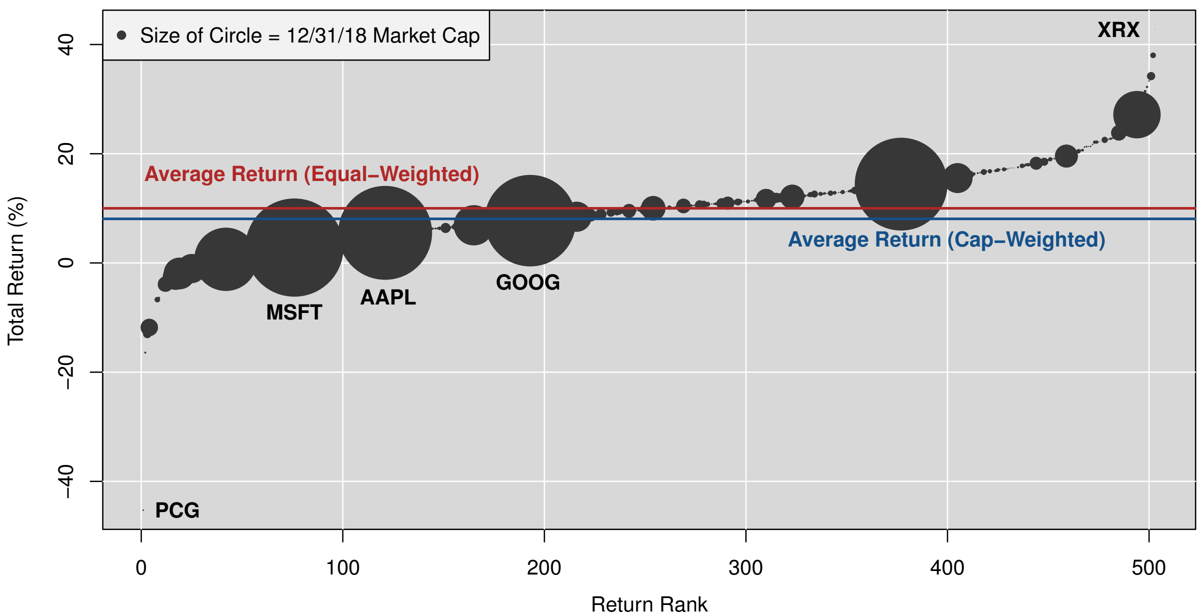

Figure 6: **Cross-Section of Individual S&P 500 Stock Returns for January 2019.** This figure shows the returns of individual stocks in the S&P 500 for the month of January 2019. The stocks are ordered from lowest return (1) to highest return (500). The companies are also represented by a circle which is proportional to their relative market capitalization as of December 31, 2018.

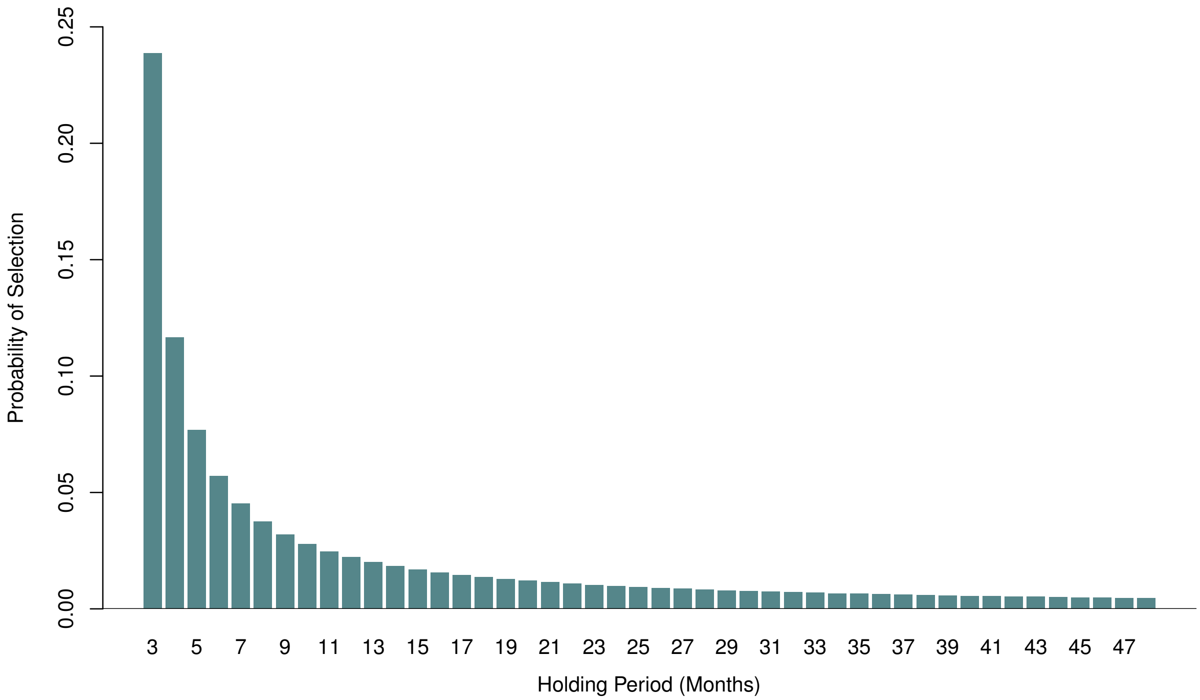

Figure 7: **Holding Period Distribution.** This figure shows the distribution of holding periods for a given stock selection by the portfolio manager. The mean holding period is twelve months, the minimum holding period is three months, and the maximum holding period is 48 months.

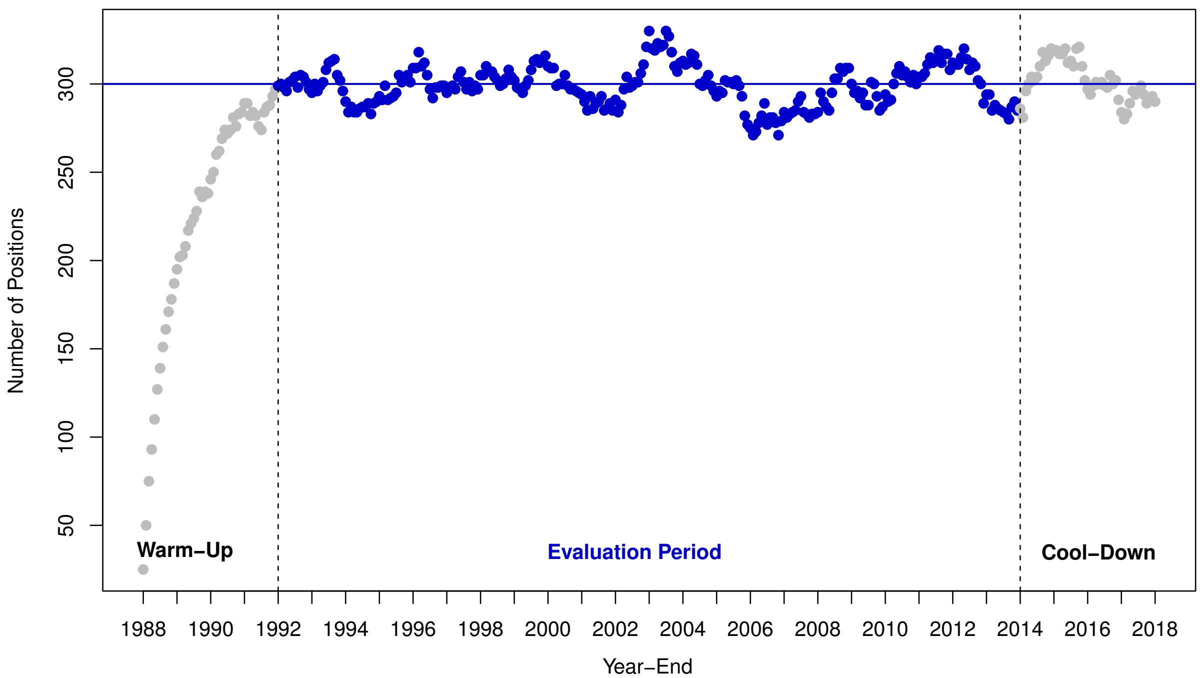

Figure 8: **Number of Stocks in a Representative Simulation Path for Dynamic Selectivity.** This figure shows the number of stocks in the portfolio from the beginning of the period (January 1988) until the final period (December 2018) for one particular simulation. The parameters for the simulation were $N = 500$, $j = 25$, $\bar{h} = 12$, and the returns are the actual returns of S&P 500 stocks for each holding period, where the holding period for any given stock is drawn from the holding period distribution.

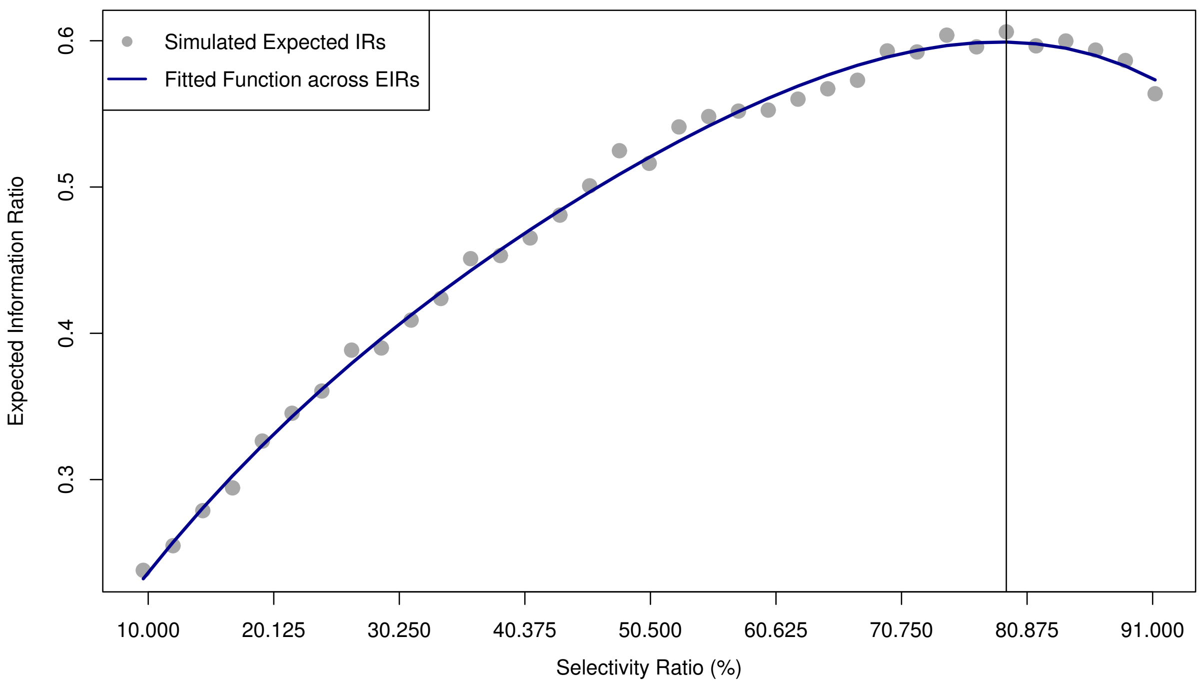

Figure 9: **Selectivity and Average information ratio for Dynamic Selectivity.** This figure shows the simulations of each selectivity ratio and its corresponding information ratio for the dynamic selectivity. The dots represent the average information ratio from XXX simulated portfolios at that particular selectivity ratio. Also shown is a fitted line across the simulation averages. For these parameters, the optimal selectivity ratio is at 80%. The parameters for the simulations are $N = 500$, $\omega = 1.1$, $j = 125$, $\bar{h} = 12$, and the returns and stocks are chosen from the actual returns and stocks of the S&P 500.

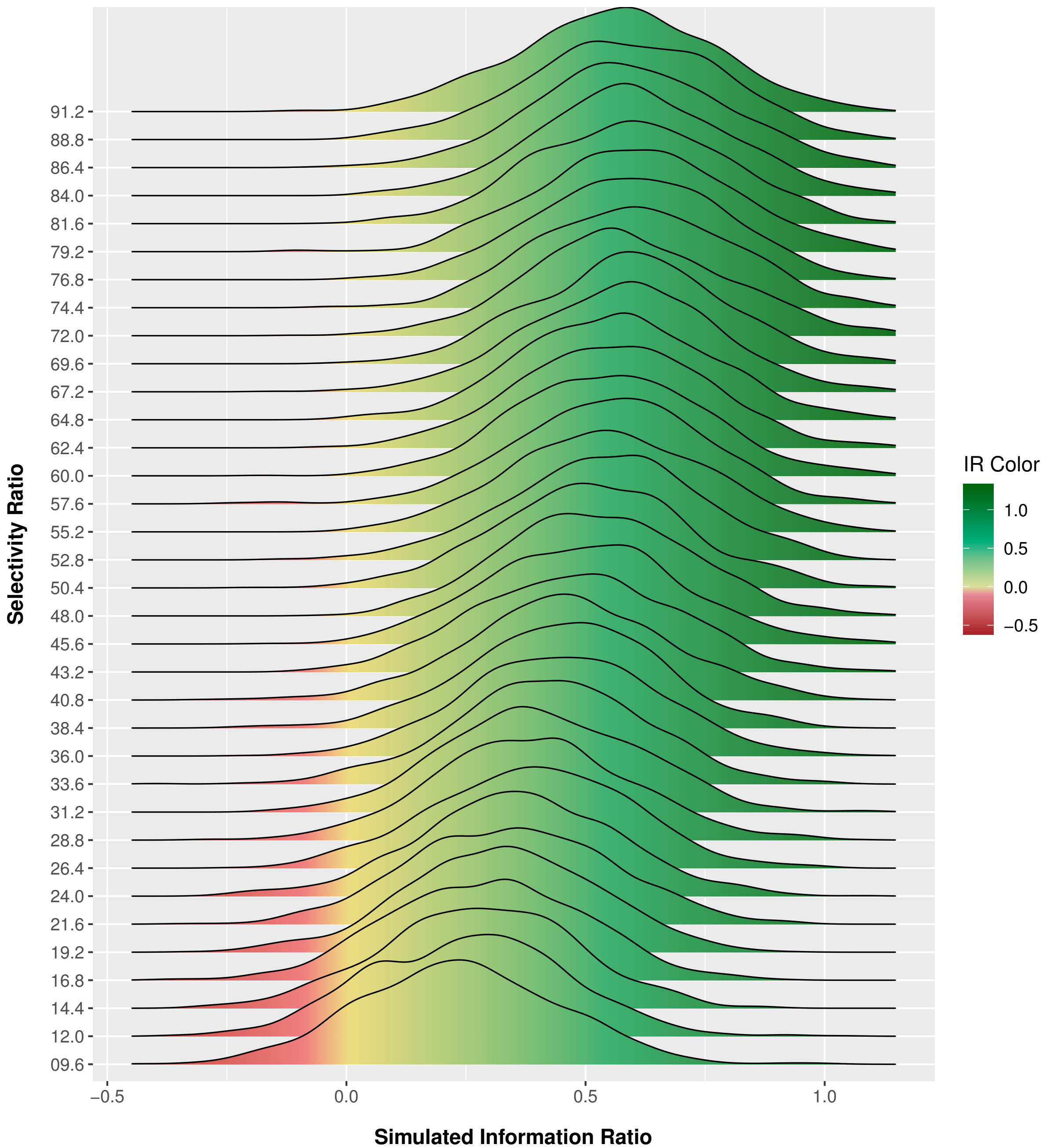

Figure 10: **Distribution of information ratios at Each Selectivity Ratio for the Simulations.** This figure shows the distribution of information ratios at each selectivity ratio for all of the 10,000 simulations. The parameters for the simulations are $N = 500$, $\omega = 1.1$, $j = 125$, $\bar{h} = 12$, and the returns and stocks are chosen from the actual returns and stocks of the S&P 500.

Appendix C   Appendix A Figures

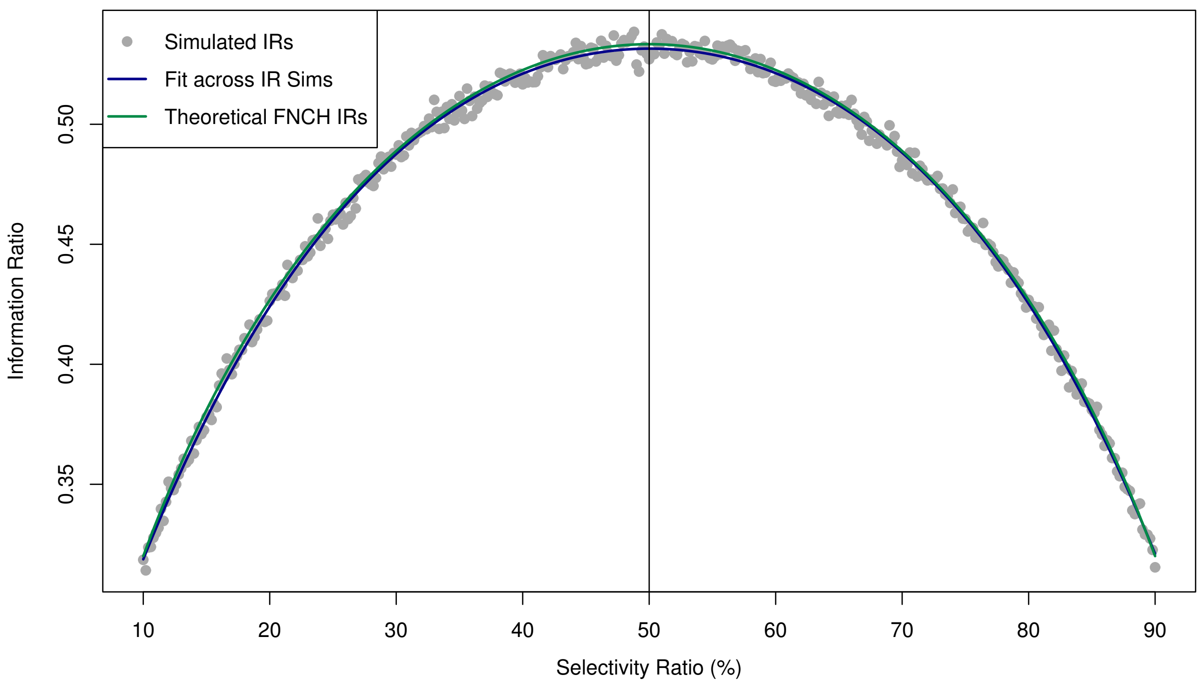

Figure 11: **Baseline Simulations: FNCH Selection Method.** This figure shows the baseline simulations of each selectivity ratio and its corresponding information ratio with the FNCH distribution method described in the Appendix. The dots represent the average information ratio from 10,000 simulated portfolios at that particular selectivity ratio. Also shown is a fitted line across the simulation averages and the threoetical Wallenius information ratios for each selectivity ratio. The optimal selectivity ratio is at 50%. The parameters for the simulations are $N = 500$, $\omega = 1.1$, $r_g = 0.10$, $r_b = -0.10$, and $r_{BM} = 0$.

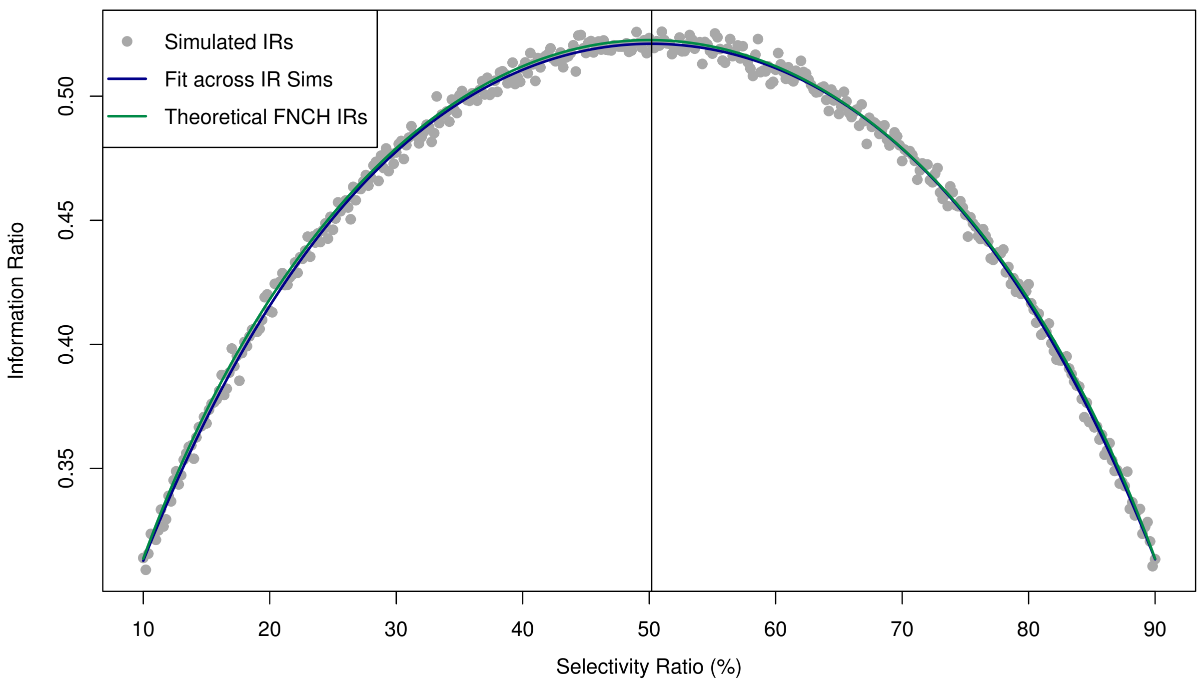

Figure 12: **The FNCH Selection Method when the Manager Only has the Ability to Find Winners in 40% of the Universe** This figure shows the results from 100,000 simulations for each selectivity level for the bulk selection method. The parameters for the simulation are $N = 500$, $\omega = 1.1$, $r_g = 10\%$, and $r_b = -10\%$. The percentage of winners in the benchmark are altered in each of the simulations. Thus, 50% corresponds to a benchmark with 250 winner stocks and 250 loser stocks, whereas 20% corresponds to a situation where the benchmark has only 100 winner stocks and 400 loser stocks and so on. The benchmark return is provided for convenience showing that when there are less than 50% winner stocks, the benchmark return is naturally negative. A.R. is for average return and I.R. is for information ratio.

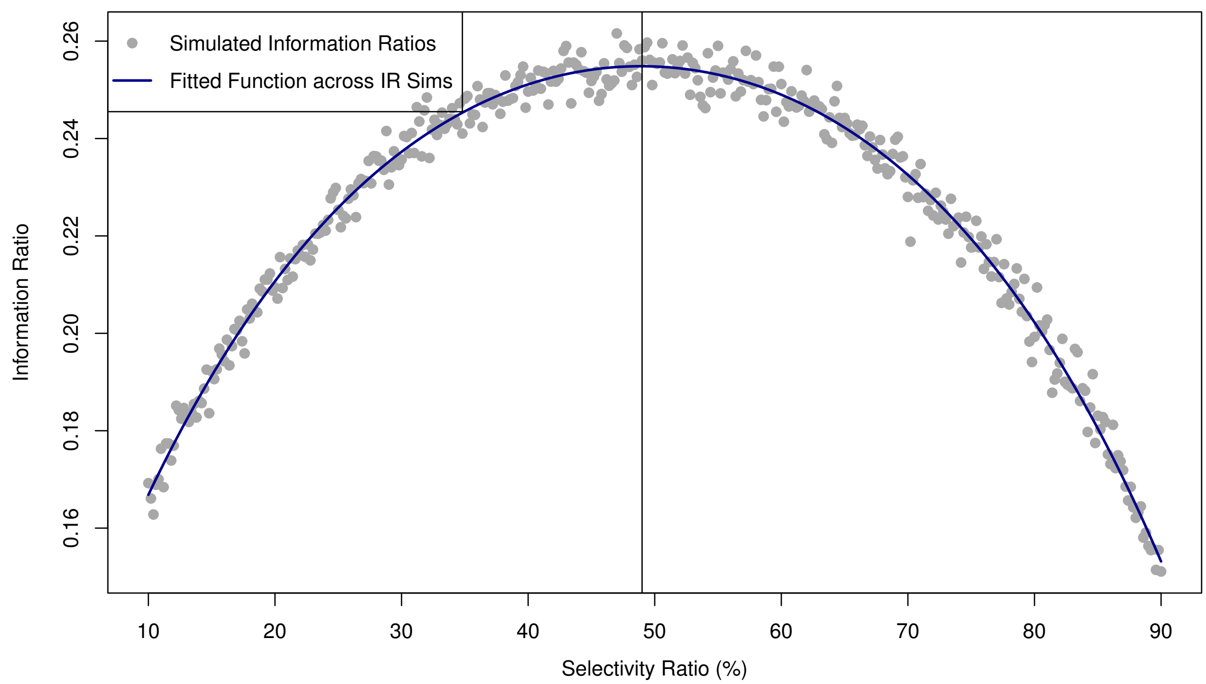

Figure 13: **Selectivity Ratio and information ratio for Empirical S&P 500 Returns for FNCH Selection Method.** This figure shows each selectivity ratio and its corresponding information ratio when returns are not binary for good and bad stocks, but are based on the cross-sectional Z-scores of actual S&P 500 individual stock returns. The dots represent the average information ratio from 10,000 simulated portfolios at that particular selectivity ratio. Also shown is a fitted line across the simulation averages and the threoetical Wallenius information ratios for each selectivity ratio. The parameters for the simulations are $N = 500$, $\omega = 1.1$, $r_i = \bar{z}_i$, and $r_{BM} = 0$.

Appendix D Appendix C: TABLES

Table 1: Portfolio Characteristics as Concentration of Winners Changes

	Percentage of Winners in Benchmark				
	50%	48%	40%	30%	20%
Benchmark Return (%)	0.000	-0.400	-2.000	-4.000	-6.000
Optimal Selectivity Ratio	79.0	79.0	79.4	79.4	79.6
A.R. at Optimal Selectivity	0.197	-0.203	-1.813	-3.839	-5.878
A.R. at 79% Selectivity	0.197	-0.203	-1.812	-3.836	-5.876
I.R. at Optimal Selectivity	0.855	0.854	0.851	0.791	0.697
I.R. at 79% Selectivity	0.855	0.854	0.846	0.795	0.694

Note: This table shows the results from 100,000 simulations for each selectivity level for the bulk selection method. The parameters for the simulation are $N = 500$, $\omega = 1.1$, $r_g = 10\%$, and $r_b = -10\%$. The percentage of winners in the benchmark are altered in each of the simulations. Thus, 50% corresponds to a benchmark with 250 winner stocks and 250 loser stocks, whereas 20% corresponds to a situation where the benchmark has only 100 winner stocks and 400 loser stocks and so on. The benchmark return is provided for convenience showing that when there are less than 50% winner stocks, the benchmark return is naturally negative. A.R. is for average return and I.R. is for information ratio.

Table 2: Portfolio Characteristics with Winner Saturation

	Percentage of Winners Manager Can Identify				
	100%	80%	60%	40%	20%
Optimal Selectivity Ratio	79.0	78.2	77.0	76.2	75.0
A.R. at Optimal Selectivity	0.197	0.161	0.124	0.084	0.042
A.R. at 79% Selectivity	0.197	0.157	0.116	0.079	0.040
I.R. at Optimal Selectivity	0.855	0.681	0.508	0.334	0.161
I.R. at 79% Selectivity	0.855	0.683	0.506	0.342	0.174

Note: This table shows the results from 100,000 simulations for each selectivity level for the bulk selection method. The parameters for the simulation are $N = 500$, $\omega = 1.1$, $r_g = 10\%$, and $r_b = -10\%$. The percentage of winners in the benchmark are altered in each of the simulations. Thus, 50% corresponds to a benchmark with 250 winner stocks and 250 loser stocks, whereas 20% corresponds to a situation where the benchmark has only 100 winner stocks and 400 loser stocks and so on. The benchmark return is provided for convenience showing that when there are less than 50% winner stocks, the benchmark return is naturally negative. A.R. is for average return and I.R. is for information ratio.

Table 3: Portfolio Characteristics when Manager Skill is Stochastic

	Standard Deviation of Skill (σ)				
	0.00	0.05	0.10	0.15	0.20
Panel A: Skill Varies for Each Stock Selection					
Optimal Selectivity Ratio	79.0	78.8	79.0	79.0	79.2
A.R. at Optimal Selectivity	0.197	0.196	0.188	0.177	0.161
A.R. at 79% Selectivity	0.197	0.196	0.188	0.177	0.163
I.R. at Optimal Selectivity	0.855	0.841	0.816	0.766	0.701
I.R. at 79% Selectivity	0.855	0.851	0.816	0.766	0.705
Panel B: Skill Varies for Each Portfolio Simulation					
Optimal Selectivity Ratio	79.0	79.0	79.0	79.2	81.0
A.R. at Optimal Selectivity	0.197	0.195	0.187	0.173	0.150
A.R. at 79% Selectivity	0.197	0.195	0.187	0.178	0.160
I.R. at Optimal Selectivity	0.855	0.788	0.627	0.474	0.354
I.R. at 79% Selectivity	0.855	0.788	0.627	0.483	0.354

Note: This table shows the results from 100,000 simulations for each selectivity level for the bulk selection method when the skill of the manager is uncertain. The parameters for the simulation are $N = 500$, $\omega \sim N(1.1, \sigma^2)$, where σ varies in the table from 0.00 to 0.20, $r_g = 10\%$, and $r_b = -10\%$. The benchmark return is 0% for information ratio calculations. The standard deviation of skill, σ, is used in two ways. In Panel A, a random draw of skill occurs before a manager selects his group of stocks for each selectivity level. In Panel B, the manager's skill is selected before each simulation of all selectivity levels. A.R. is for average return and I.R. is for information ratio.

Table 4: Summary Statistics about S&P 500 Data

	Market Capitalization Weighted								**Equal Weighted**						
Time Period	μ	σ	Sharpe	Max	Min	% Winners	Herf.	μ	σ	Sharpe	Max	Min	% Winners	Herf.
1989-2018	0.88	4.08	0.16	11.44	-16.69	0.50	79.07	1.00	4.63	0.16	18.48	-20.91	0.49	20.00
1989-1998	1.54	3.85	0.29	11.44	-14.37	0.48	71.67	1.37	4.17	0.23	11.38	-14.88	0.49	20.00
1999-2008	-0.01	4.35	-0.06	9.93	-16.69	0.51	90.27	0.35	5.08	0.02	11.04	-20.91	0.49	20.00
2009-2018	1.11	3.91	0.28	10.89	-10.40	0.50	75.27	1.28	4.56	0.27	18.48	-11.74	0.50	20.00

Note: This table shows summary statistics of the S&P 500 stocks used in the analysis of this paper. μ represents the monthly average return of the stocks over the relevant sample period, σ represents the standard deviation of the monthly returns of the stocks over the sample period, Sharpe represents the Sharpe ratio for the portfolio of stocks over the sample period, Max represents the maximum monthly return for an individual stock over the sample period, Min represents the minimum monthly return for an individual stock over the sample period, % Winners is the number of stocks that have a monthly return above the average of the benchmark, and Herf. represents the herfindahl index of diversification of stocks with 20 being the most diversified and 100 being the most undiversified.

Table 5: Simulation Results for Market Capitalization Weighted Benchmarks

	Type of Market Cap Benchmark				
	Strong Large-Cap Tilt	Large-Cap Tilt	Neutral Size	Small-Cap Tilt	Strong Small-Cap Tilt
Panel A: Equally Weighted Portfolios					
Optimal Selectivity Ratio	10.0	10.0	78.8	90.0	90.0
A.R. at Optimal Selectivity	0.448	0.449	0.198	0.122	0.121
A.R. at 79% Selectivity	0.196	0.198	0.197	0.196	0.197
I.R. at Optimal Selectivity	-5.071	-0.595	0.861	9.246	48.947
I.R. at 79% Selectivity	-30.138	-4.559	0.851	6.260	31.848
Panel B: Market Capitalization Weighted Portfolios					
Optimal Selectivity Ratio	82.6	78.0	78.4	78.6	74.2
A.R. at Optimal Selectivity	7.231	1.448	0.203	-1.050	-7.032
A.R. at 79% Selectivity	7.241	1.444	0.201	-1.055	-7.046
I.R. at Optimal Selectivity	0.586	0.397	0.407	0.406	0.611
I.R. at 79% Selectivity	0.586	0.404	0.411	0.401	0.620
Benchmark Return	7.151	1.250	0.000	-1.250	-7.151

Note: This table shows the results from 100,000 simulations for each selectivity level for the bulk selection method using a market capitalization weighted index. The parameters for the simulation are $N = 500$, $\omega = 1.1$, $r_g = 10\%$, and $r_b = -10\%$. The portfolios are compared against five types of market capitalization benchmark, one in which all winner stocks are the largest 250 stocks in the index ("Strong Large-Cap Tilt"), one in which all of the winner returns are the smallest 250 stocks in the index ("Strong Small-Cap Tilt"), one in which the returns are distributed such that the market capitalization weighted returns are equal to 0 (i.e. $\left(\sum_{k=1}^{n_g} MC_k \approx \sum_{j=1}^{n_b} MC_j\right)$), and one which winners are slightly tilted towards big stocks ("Large-Cap Tilt") and one in which winner stocks are slightly tilted towards smaller companies ("Small-Cap Tilt"). Two type of portfolios are analyzed with respect to selectivity; an equally weighted portfolio and a market capitalization weighted portfolio. A.R. is for average return and I.R. is for information ratio.

Table 6: CashFlow-to-Price Factor and Selectivity

Panel A: Individual Decile Performance										
	1	2	3	4	5	6	7	8	9	10
Average Return	19.25	18.03	16.57	16.56	15.28	14.87	14.17	13.51	12.43	11.02
Marginal Decline in A.R.	-	1.22	1.46	0.01	1.28	0.41	0.71	0.66	1.08	1.40
Panel B: Cumulative Decile Performance										
	1	2	3	4	5	6	7	8	9	10
Average Return	19.25	18.64	17.95	17.60	17.14	16.76	16.39	16.03	15.63	15.17
Excess Return	4.08	3.47	2.78	2.43	1.97	1.59	1.22	0.86	0.46	0.00
Tracking Error	5.86	4.41	3.39	2.84	2.39	2.01	1.67	1.34	0.85	0.00
information ratio	0.70	0.79	0.82	0.86	0.82	0.79	0.73	0.64	0.54	-

Note: This table shows the historical returns from July 1951 to May 2019 of the difference deciles of the Fama-French cash flow-to-price variable. Average return is the annualized average return of the decile over the entire period. Excess return is the return of the particular decile, or deciles in the case of the cumulative measure minus the returns of the benchmark, which are the entire universe of stocks. Tracking Error is ...

Table 7: Portfolio Characteristics of Dynamic Portfolio Selection

Number of Monthly Stock Purchases	17	21	25	26	27	33	37
Average Selectivity Ratio (%)	40.8	50.4	60.0	62.4	64.8	79.2	88.8
Panel A: IR Expected Returns							
Cross-Sectional Mean (“EIR”)	0.440	0.467	0.484	0.481	0.482	0.450	0.387
Polynomial Fit across EIRs	0.440	0.467	0.481	0.482	0.482	0.450	0.387
Panel B: information ratios							
Average IR							
Median IR	0.439	0.469	0.484	0.477	0.480	0.445	0.385
Maximum IR	1.313	1.298	1.234	1.318	1.479	1.445	1.162
Minimum IR	-0.414	-0.318	-0.338	-0.296	-0.238	-0.230	-0.408

Note: This table shows the results from 10,000 simulations for a different number of stocks acquired by the portfolio manager each month. The parameters for the simulation are $N = 500$, $\omega = 1.1$, $\bar{h} = 12$, and stock returns are obtained from actual S&P 500 returns. I.R. is the information ratio.

Table 8: Portfolio Characteristics of Dynamic Selectivity with Altered Parameters

	Skill Parameters			Average Holding Period (j)		
Skill (ω)	1.05	1.10	1.20	1.10	1.10	1.10
Average Holding Period (j)	12	12	12	1	12	24
Number Simulations	10,000	10,000	10,000	10,000	10,000	10,000
Panel A: Sequential Selection						
Optimal Selectivity Ratio (%)	79.2	79.2	79.2	79.2	79.2	76.8
IR at Optimal Selectivity	0.353	0.599	1.050	1.973	0.599	0.444
Panel B: Bulk Selection						
Optimal Selectivity Ratio (%)	55.2	55.2	55.2	50.0	55.2	57.6
IR at Optimal Selectivity	0.273	0.482	0.870	1.237	0.482	0.383

Note: This table shows the results from 10,000 simulations for a different number of stocks acquired by the portfolio manager each month. The parameters for the simulation are $N = 500$ and the parameters listed in the table, and stock returns are obtained from actual S&P 500 returns.